Model Predictive Optimal Averaging Level Control

The infinite-horizon, discrete-time optimal averaging level control problem for surge tanks, with minimization of the rate of change of outlet flow as its objective, is formulated and a solution is presented. A finite moving-horizon approximation is introduced and analytical solutions are obtained for two important special cases. These results provide a quantitative measure of the impact of a secondary objective, integral action, on flow filtering. The problem is then generalized to include nonconstant level and outlet flow constraints. A model predictive control formulation is presented which addresses the objectives of the generalized problem. The resulting controller minimizes the maximum rate of change of outlet flow, provides integral action, and handles constraints on the tank level and outlet flow rate. The proposed controller includes a single adjustable parameter that directly effects the trade-off between the incompatible objectives of good flow filtering and rapid settling time. Examples are presented to demonstrate the properties of the model predictive controller. An implementation, involving imbedded feedback, is developed which guarantees internal stability of the model predictive scheme for open-loop unstable processes (such as integrators).

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Introduction

In model predictive control (MPC) future system inputs are selected which optimize a performance objective over a finite future time horizon, subject to system constraints. The performance objective is generally a weighted measure of future tracking error, the difference between predicted outputs and desired set points. In each time step, an estimate of current disturbances is updated, the optimization is solved based on this new estimate, and the first of the resulting optimal future inputs is implemented. Updating the disturbance estimate and solving the optimization at each time step compensates for unmeasured disturbances and model inaccuracies (which cause actual system outputs to be different from the predicted outputs). Many objetive functions and system constraints result in optimization problems that can be formulated as linear or quadratic programs (Campo and Morari, 1986), for which efficient and reliable solution techniques exist. Several such schemes have been advanced in the last ten years. These include, among others, model algorithmic control (Richalet et al., 1978), dynamic matrix control (Cutler and Ramaker, 1979), and internal model control (Garcia and Morari, 1982). The most significant feature of these control algorithms is their ability to handle system constraints in an optimal fashion. In this paper we apply these ideas to the solution of the so-called optimal averaging level control problem (McDonald et al., 1986).

The objective in surge tank control is to effectively use the tank capacity to filter inlet flow disturbances and prevent their propagation to downstream units. Tight control around a specific level set point is usually unnecessary and is contrary to the flow disturbance filtering objective. Tank level and outlet flow constraints, however, must not be violated and it is desirable to eventually return the tank inventory to its nominal value so that capacity is available to filter future flow disturbances. Since set-point tracking and rapid integral action are only secondary objectives, level constraints dominate the problem. Indeed, if the tank had infinite capacity there would be no need for control; the outlet flow could be held constant, achieving perfect flow filtering.

The flow filtering objective is quantified by the maximum rate of change of outlet flow (MRCO) for a given inlet flow disturbance. Traditionally, this objective has been achieved by using proportional or proportional-integral level control, sufficiently detuned to provide reasonable flow filtering (Cheung

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and Luyben, 1979) or by simple, intuitively based nonlinear schemes (Kutten et al., 1972). More recently, an optimal averaging strategy has been advanced (McDonald et al., 1986). This approach, which directly addresses the objective of optimal flow filtering subject to level constraints, has much appeal. The integral nature of constraints in this optimal strategy suggests a model predictive implementation.

Continuous-Time Optimal Averaging Level Control

In this section we present a summary of the work of McDonald et al. (1986), who first presented control schemes to directly address the flow filtering objective. Using a generalization of the derivative of outlet flow (MRCO) the flow filtering objective is defined:

$$\min_{q_0(t)} \sup_{\substack{t,t' \in (0,\infty) \\ t \neq t'}} \left| \frac{q_0(t) - q_0(t')}{t - t'} \right| \tag{1}$$

Subject to:

$$h_{min} \le h(t) \le h_{max} \quad t \in (0, \infty) \tag{2}$$

where $q_o(t)$ is the tank outlet flow at time t, and $h(t)$ is the tank level at time t.

While the MRCO objective, Eq. 1, addresses the primary flow filtering objective, it does not address a number of secondary objectives. For example a solution to Eqs. 1 and 2 need not provide integral action and might result in outlet flows and tank levels that are excessively oscillatory. Indeed as we will see, Eqs. 1 and 2 admit an infinite number of solutions, many of which have one or more of these undesirable properties. Nonetheless, by focusing attention directly on the flow filtering objective, the synthesis of controllers for this objective (as opposed to tuning rules for controllers with some prescribed structure), and by providing a meaningful performance measure for analysis, the MRCO objective is very useful.

Solutions to Eqs. 1 and 2 depend upon the nature of the expected inlet flow disturbances. For a step disturbance of magnitude B entering a tank of constant area A, at its nominal condition of $q_{os} = 0$, $h_s = 0$, McDonald et al. show that Eqs. 1–2 admit the following solution:

$$q_o(t) = \begin{cases} \dfrac{B^2 t}{2Ah_{lim}} & t \in (0, t^*] \\ B & t > t^* \end{cases} \tag{3}$$

where:

$$t^* = \frac{2A|h_{lim}|}{B}$$

$$h_{lim} = \begin{cases} h_{max} \text{ if } B > 0 \\ h_{min} \text{ if } B < 0 \end{cases}$$

For a given MRCO, the most effective way to increase (or decrease) $q_o(t)$ to offset the flow imbalance, $B - q_o(t)$, is to increase (or decrease) $q_o(t)$ at a constant rate. It is easy to verify that the solution, Eq. 3, is a ramp of minimum slope that completely offsets the flow imbalance just as the level reaches its limit (at time t^*). A ramp of lower slope would allow the level limit to be exceeded before the flow imbalance is eliminated (and would therefore be infeasible); a ramp of greater slope would drive the imbalance to zero before the level reached its limit (and would therefore be nonoptimal). Thus the solution is unique for $t \in (0, t^*]$. For $t > t^*$ any $q_o(t)$ which satisfies:

$$\left| \frac{q_o(t) - q_o(t')}{t - t'} \right| \le \frac{B^2}{2A|h_{lim}|} \; \forall t \neq t', \quad t > t^* \tag{4}$$

and

$$B(t - t^*) - A(h_{max} - h_{lim}) \le \int_{t^*}^{t} q_o(t)\, dt \le B(t - t^*) - A(h_{min} - h_{lim}) \tag{5}$$

is also optimal. These conditions simply insure that for $t > t^*$, the rate of change of outlet flow is less than that for $t \in (0, t^*]$, Eq. 4, and that the level constraints are not violated for $t > t^*$, Eq. 5. In keeping with the desire to keep the outlet flow constant subject to level constraints, it is set equal to the inlet flow, B, for $t > t^*$ to arrive at the solution, Eq. 3. Since the supremum in Eq. 1 is taken over all future time, we will refer to Eq. 3 as the infinite-horizon solution.

As shown by McDonald et al., the infinite-horizon solution can be implemented as a nonlinear proportional feedback. However, in order to insure that level constraints are not violated, B must equal the magnitude of the largest anticipated step disturbance. This results in suboptimal performance for disturbances of lesser magnitude. Indeed, in this scheme MRCO is independent of the magnitude of the disturbances that are realized. Additionally, since proportional feedback cannot eliminate steady-state offset, an integral term, detuned to minimize impact on the optimal MRCO, is added. This detuned integral action provides a slow return to the nominal level. Should additional disturbances occur before the nominal condition is attained, level constraints could be violated.

When measurements of the inlet flow disturbances are available, McDonald et al. propose a feedforward/feedback scheme whose response is dependent on the magnitude of the measured disturbance. Again, in order to eliminate steady-state offset, proportional and integral modes are added to the MRCO optimal controller with an associated increase in MRCO. This optimal predictive controller (OPC) is defined by:

$$q_o = \tilde{q}_o + K_c h + \frac{K_c}{\tau_i} \int h\, dt \tag{6a}$$

$$\dot{\tilde{q}}_o = \frac{(q_i - \tilde{q}_o)^2}{2A(h_{lim} - h)} \tag{6b}$$

where:

$\tilde{q}_o$ = MRCO optimal outlet flow rate

$q_i(t)$ = measured inlet flow rate

K_c = proportional gain

τ_i = integral reset time

$\dot{x} = dx/dt$

While this formulation provides better flow filtering for small disturbances (of magnitude less than the maximum anticipated), it requires measurement of the inlet flow rate and integral action is achieved at the expense of MRCO optimality.

Discrete-Time Optimal Flow Filtering

With outlet flow constant between sample times, the discrete-time, infinite-horizon optimal flow filtering problem can be expressed, at time t, as:

$$\min_{q_o(t+k)} \max_{k \in K} |q_o(t + k) - q_o(t + k - 1)| \tag{7}$$

Subject to:

$$h_{min} \leq h(t + k + 1) \leq h_{max} \quad k \in K \tag{8}$$

where $K = \{0, 1, 2, \ldots\}$.

Defining nominal conditions $q_{os} = 0$, $h_s = 0$, a simple mass balance provides,

$$h(t + k + 1) = h(t) - \frac{T}{A} \sum_{j=0}^{k} [q_o(t + j) - d(t + j)] \tag{9}$$

where $d(t)$ is the inlet flow disturbance realized at time t, and T is the sampling time.

Since exact prediction of the future level using Eq. 9 requires knowledge of current and future inlet flow disturbances, $[d(t), d(t + 1) \ldots]$, we cannot solve Eq. 7 subject to Eq. 8. Instead we will make assumptions that allow us to predict the future level based on currently available information and solve instead,

$$\min_{q_o(t+k)} \max_{k \in K} |q_o(t + k) - q_o(t + k - 1)| \tag{10}$$

Subject to:

$$h_{min} \leq h(t + k + 1 | t) \leq h_{max} \quad k \in K \tag{11}$$

where the notation $h(t + k | t)$ indicates the estimate of h at time $t + k$ based on information available at time t.

We now turn to the assumptions that allow us to evaluate $h(t + k + 1 | t)$. In this formulation we use a model of the plant to infer inlet flow disturbances. With this approach it is unnecessary to measure the inlet flow rate explicitly. The formulation is based on the internal model control (IMC) structure of Garcia and Morari (1982), shown in Figure 1.

The effect of inlet flow disturbances, d, on the output, h, is evaluated at each sample time by subtracting the model output, $\tilde{h}(t)$, from the measured output, $h(t)$:

$$d_h(t) = h(t) - \tilde{h}(t) \tag{12}$$

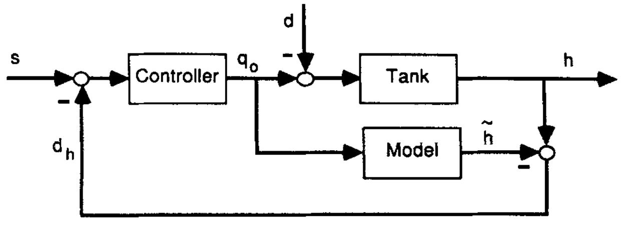


Figure 1. Internal model control (IMC) structure.

It is assumed that d_h represents only the effects of unmeasured disturbances on the output (although it includes the effects of modeling errors as well). The internal model, relating tank level to outlet flow is

$$\tilde{h}(t) = \tilde{h}(t - 1) - \frac{T}{A} q_o(t - 1) \tag{13}$$

so that

$$d_h(t) - d_h(t - 1) = h(t) - \tilde{h}(t) - h(t - 1) + \tilde{h}(t - 1) \tag{14a}$$

$$= h(t) - h(t - 1) + \frac{T}{A} q_o(t - 1) \tag{14b}$$

The assumption that d_h represents only the effects of inlet flow disturbances allows us to use $d_h(t)$ to estimate the inlet flow disturbance realized at time $t - 1$. From Eq. 9 we have

$$d(t - 1) = \frac{A}{T} [h(t) - h(t - 1)] + q_o(t - 1) \tag{15}$$

and from Eq. 14b we see that the righthand side of Eq. 15 is equal to $(A/T)[d_h(t) - d_h(t - 1)]$. Thus our estimate of the inlet flow disturbance that occurred at time $t - 1$ is

$$d(t - 1 | t) = \frac{A}{T} [d_h(t) - d_h(t - 1)] \tag{16}$$

Since the optimal averaging level control problem was originally defined for step inlet flow disturbances, we assume that any inlet flow disturbance that occurred at time $t - 1$ (the most recent we can detect using level measurements) is constant. With the further assumption that no new disturbances will enter at time t or in the future, we have

$$d(t + k | t) = d(t - 1 | t) = \frac{A}{T} [d_h(t) - d_h(t - 1)] \tag{17}$$

Using this result and Eq. 9, the prediction of future level is given by:

$$h(t + k + 1 | t) = h(t) + (k + 1) \frac{T}{A} d(t + k | t) - \frac{T}{A} \sum_{j=0}^{k} q_o(t + j) \tag{18}$$

and our definition of the discrete-time optimal flow filtering problem, Eqs. 10–11 is complete.

With the following theorem we characterize all solutions to this problem. In order to simplify the notation we define,

$$\Omega(t) = d(t - 1 | t) - q_o(t - 1) \tag{19}$$

the estimated flow imbalance, inlet flow minus outlet flow, at time t_-, immediately before we implement $q_o(t)$.

Theorem 1. The sequence $q_o(t + k)$, $k \in K$ is a solution to the discrete-time, infinite-horizon optimal flow filtering problem, Eqs. 10, 11, and 18, if and only if:

1. $q_o(t + k) = q_o(t + k - 1) + \Delta q_o^* \quad \forall\, k \in [0, k^*)$

2. $h_{max} - h_{lim} \geq (T/A)\, \Sigma_{k=k^*}^{j} [d(t+k|t) - q_o(t+k)] \geq h_{min} - h_{lim} \quad \forall\, j \geq k^*$

3. $|q_o(t+k) - q_o(t+k-1)| \leq \Delta q_o^* \quad \forall\, k \geq k^*$

where:

$$\Delta q_o^* = \frac{2\Omega(t)}{(k^*+1)} - \frac{2A[h_{lim} - h(t)]}{Tk^*(k^*+1)} \tag{20}$$

$$k^* = N\left\{\frac{2A[h_{lim} - h(t)]}{T\Omega(t)}\right\} \tag{21}$$

$$h_{lim} = \begin{cases} h_{max} \text{ for } \Omega(t) > 0 \\ h_{min} \text{ for } \Omega(t) < 0 \end{cases}$$

and $N\{x\}$ indicates the smallest integer $\geq x$.

Proof. See appendix A.

Condition 1 specifies that the solution is a ramp change in outlet flow that completely offsets the flow imbalance just as the level reaches its limit (at time $t + k^*$). As one might expect, k^* decreases as the magnitude of the flow imbalance increases. As for the continuous case, the optimal discrete-time solution is nonunique (for $t \geq t + k^*$). Conditions 2 and 3 are analogous to Eqs. 5 and 4 from the continuous case and characterize admissible outlet flow rates for $t \geq t + k^*$. In particular, condition 3 insures that outlet flow rate changes for $t \geq t + k^*$ are smaller than for $t < t + k^*$. Condition 2 insures that the level constraints are not violated for $t \geq t + k^*$.

A particular solution to Eqs. 10, 11, and 18 is provided by:

$$q_o(t+k) = \begin{cases} q_o(t+k-1) + \Delta q_o^* & k \in [0, k^*) \\ d(t+k|t) & k \geq k^* \end{cases} \tag{22}$$

where we have resolved the nonuniqueness by making the outlet flow constant for $k \geq k^*$.

The MRCO given by Eq. 22 is:

$$MRCO^* = \frac{2\Omega(t)}{T(k^*+1)} - \frac{2A[h_{lim} - h(t)]}{T^2k^*(k^*+1)} \tag{23}$$

Since

$$k^* \geq \frac{2A[h_{lim} - h(t)]}{T\Omega(t)} \tag{24}$$

where equality holds when the righthand side is an integer, we have (substituting Eq. 24 into Eq. 23):

$$MRCO^* \geq \frac{\Omega^2(t)}{2A[h_{lim} - h(t)] - T\Omega(t)} \tag{25}$$

The MRCO provided by the continuous-time, infinite-horizon solution, Eq. 3, is given by:

$$MRCO = \frac{B^2}{2A[h_{lim} - h(t)]} \tag{26}$$

Noting that B is the flow imbalance, $\Omega(t)$, in the continuous case, and comparing Eq. 26 with Eq. 25, we find an additional term in the denominator of Eq. 25 due to the lag of T time units (one sample time) required to infer the flow imbalance using level measurements. In the discrete implementation, we can only adjust the outlet flow at the sample times and this causes MCRO to be greater than the bound in Eq. 25 in general (whenever the righthand side of Eq. 24 is not an integer).

While this formulation is useful for discrete-time level control when feedforward measurements are not available, it has several drawbacks. Most significantly, there is no provision for integral action. The ramp solution, Eq. 22, allows the tank level to move to its constraint and remain there indefinitely. Subsequent disturbances result in immediate level constraint violation. In addition, no consideraton has been given to the effects of outlet flow rate constraints. In the subsequent sections we will show how these issues can be addressed in the framework of model predictive control.

Model Predictive Formulation

In this section we will develop the optimal flow filtering objective as a model predictive control problem. This formulation is based on a finite-horizon analog of the discrete-time flow filtering problem, Eqs. 10, 11, and 18. As we will show in the next section, the optimization can be recast as a linear program to be solved on-line. As is standard in model predictive control, only the first of the optimal future inputs is implemented and the optimization is resolved at each sample time.

With the restriction of a finite future horizon, P sample times in length, we can write Eqs. 10, 11, and 18 as:

$$\min_{q_o} \|\boldsymbol{R}\boldsymbol{q}_o - e_1 q_o(t-1)\|_\infty \tag{27}$$

Subject to:

$$\boldsymbol{1}h_{min} \leq -\boldsymbol{H}\boldsymbol{q}_o + \boldsymbol{n}\frac{T}{A}d(t+k|t) + \boldsymbol{1}h(t) \leq \boldsymbol{1}h_{max} \tag{28}$$

where: $\|x\|_\infty = \max_i |x_i|$ is the $\infty-$ norm on $\mathcal{R}^P$.

$$\boldsymbol{q}_o = \begin{pmatrix} q_o(t) \\ q_o(t+1) \\ \cdot \\ \cdot \\ q_o(t+P-1) \end{pmatrix} \quad \boldsymbol{R} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \boldsymbol{n} = \begin{pmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ P \end{pmatrix} \quad \boldsymbol{1} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \quad \boldsymbol{H} = \begin{pmatrix} \frac{T}{A} & 0 & \cdots & 0 \\ \frac{T}{A} & \frac{T}{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{T}{A} & \frac{T}{A} & \cdots & \frac{T}{A} \end{pmatrix}$$

Note that $\boldsymbol{H}$ is the truncated impulse response matrix for the tank.

It should be stressed that $d(t + k|t)$, the inferred inlet flow disturbance, is reevaluated at each sample time and the optimization is resolved based on the new information. This approach mitigates the consequences of the restrictive assumptions made about future disturbances.

In the MPC framework we are free to impose constraints more general than Eq. 28. Constraints on any linear combination of future inputs and outputs can be handled by the on-line optimization, (nonlinear constraints preclude the use of linear programming to solve the optimization). In particular, upper and lower bounds on level can be specified at each future time step independently. Constraints can also be specified for the manipulated variable, insuring that the control algorithm will not demand outlet flow rates that exceed actuator saturation. Defining α_i, β_i, as lower and upper bounds on the outlet flow rate at time $t + i - 1$, and γ_i, δ_i, as lower and upper bounds on level at time $t + i$, we can generalize Eqs. 27–28 as:

$$\min_{q_o} \|\boldsymbol{R}\boldsymbol{q}_o - e_1 q_o(t-1)\|_\infty \tag{29}$$

subject to:

$$-\boldsymbol{q}_o \leq -\boldsymbol{\alpha}$$
$$\boldsymbol{q}_o \leq \boldsymbol{\beta} \tag{30a}$$

$$\boldsymbol{H}\boldsymbol{q}_o \leq -\boldsymbol{\gamma} + \boldsymbol{n}\frac{T}{A}d(t+k|t) + \mathbf{1}h(t)$$
$$-\boldsymbol{H}\boldsymbol{q}_o \leq \boldsymbol{\delta} - \boldsymbol{n}\frac{T}{A}d(t+k|t) - \mathbf{1}h(t) \tag{30b}$$

Usually the outlet flow rate constraints are specified by the capabilities of process equipment and are the same in each future time step, that is, $\boldsymbol{\alpha} = \mathbf{1}q_{omin}$, $\boldsymbol{\beta} = \mathbf{1}q_{omax}$. For multiple tasks in series it may be desirable to define q_{omin}, q_{omax} to prevent large flow overshoot, which is magnified by each tank in the series (Cheung and Luyben, 1979). Nonconstant future level constraints can be used to prescribe a future level trajectory that has certain desired properties such as zero offset at some future time. Two simple forms of level constraints, for which we can obtain analytical solutions to Eqs. 29–30 in the absence of outlet flow constraints, (i.e., neglecting Eq. 30a) will be discussed in detail.

Constant level constraints

To evaluate the impact of the finite horizon on flow filtering we first define level constraints that are constant over the future horizon as in the infinite-horizon problem. Specifically we have:

$$\boldsymbol{\gamma} = \mathbf{1}h_{max}$$
$$\boldsymbol{\delta} = \mathbf{1}h_{min} \tag{31}$$

The outlet flow given by solving Eqs. 29, 30, and 31 and implementing the first element of $\boldsymbol{q}_o^*$ at each sample time is:

$$q_o(t) = \begin{cases} q_o(t-1) & P \leq \dfrac{k^*}{2} \\ q_o(t-1) + \Delta q_o^P & \dfrac{k^*}{2} < P < k^* \\ q_o(t-1) + \Delta q_o^* & P \geq k^* \end{cases} \tag{32}$$

where

$$\Delta q_o^P = \frac{2\Omega(t)}{P+1} - \frac{2A[h_{lim} - h(t)]}{TP(P+1)} \tag{33}$$

and k^* is as defined in Eq. 21.

While notationally involved, this solution is easy to understand. Disturbances that result in $k^*/2 \geq P$ are of sufficiently small magnitude that even if no outlet flow changes are made, the predicted level will remain within the minimum and maximum bounds over the future horizon. The optimal solution is to keep the outlet flow constant. For larger disturbances, which result in $k^*/2 < P < k^*$, a nonzero change in outlet flow is needed to insure that the level does not violate its bound. Since the flow imbalance will never change sign, the level changes monotonically in time, and it is sufficient to insure that the level is not violated at the end of the time horizon. Δq_o^P is the smallest constant outlet flow change that satisfies this condition. For large disturbances $k^* \leq P$, the predicted future level will reach its bound at time k^*, and the optimal outlet flow change is the same as in the infinite-horizon case, Δq_o^*. This provides several insights. For a disturbance observed at time t (i.e., $\Omega(t) \neq 0$) if $P \geq k^*$ the infinite-horizon MRCO optimal solution, Eq. 22, is realized. Thus filtering of flow imbalances, $\Omega(t)$, whose magnitude is greater than $2A[h_{lim} - h(t)]/TP$ is not impaired by the finite-horizon restriction. Equivalently, it is possible to achieve optimal flow filtering of arbitrarily small flow imbalances by selecting P adequately large.

As in the infinite-horizon case, integral action is not provided. The level moves to its constraint and remains there in response to an arbitrarily small step inlet disturbance. This lack of integral action prevents this formulation from being useful in any practical situation. As in the definition of the OPC, an integral term could be added to the optimal solution. This approach results in an increase in MRCO that is difficult to quantify. There is no clear method for selecting an integral reset time that yields a good trade-off between the incompatible objectives of small settling time and small MRCO. In the following we show how a modification of the level constraints provides both integral action *and* MRCO optimal filtering of disturbances of magnitudes above a specified threshold.

Box level constraints

Modifying the constant level constraints to include the fixed end point condition, $h(t + P|t) = 0$, results in:

$$\boldsymbol{\gamma} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \\ 0 \end{pmatrix} h_{min} \quad \boldsymbol{\delta} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ 1 \\ 0 \end{pmatrix} h_{max} \tag{34}$$

which we will refer to as box constraints.

With these constraints the optimization finds, at each sample time, future outlet flow rates for which the predicted level reaches its nominal value in P time steps. The outlet flow given by solving Eqs. 29, 30b, and 34 at each sample time is given by:

$$q_o(t) = \begin{cases} q_o(t-1) + \Delta q_o^0 & |\Delta q_o^0| > |\Delta q_o^*| \\ q_o(t-1) + \Delta q_o^* & |\Delta q_o^0| \le |\Delta q_o^*| \end{cases} \tag{35}$$

where:

$$\Delta q_o^0 = \frac{2\Omega(t)}{P+1} + \frac{2A[h(t) - h_s]}{TP(P+1)} \tag{36}$$

and Δq_o^* is as defined in Eq. 20.

As in the constant level constraint case, this solution has a straightforward interpretation. Δq_o^* is the minimum magnitude change in outlet flow that prevents constraint violation for times less than $t + k^*$; Δq_o^0 is the minimum magnitude change in outlet flow that satisfies the fixed end-point condition. The best feasible solution is then clearly the larger of these flow changes. For a particular choice of P, large flow imbalances result in $|\Delta q_o^0| < |\Delta q_o^*|$ and the solution $q_o(t) = q_o(t-1) + \Delta q_o^*$ is implemented. Since this recovers the discrete-time, infinite-horizon MRCO optimal solution, Eq. 22, the fixed end-point condition has no effect on filtering performance. For small imbalances, $|\Delta q_o^0| > |\Delta q_o^*|$ and the solution $q_o(t) = q_o(t-1) + \Delta q_o^0$ is implemented. In this situation the fixed end-point condition causes an increase in MRCO. Theorem 2 provides a condition on the horizon length which insures that the fixed end-point condition does not interfere with flow filtering.

Theorem 2. For step inlet flow disturbances, the sequence $q_o(t+k)$, $k \in K$ determined by Eq. 35 satisfies the conditions of theorem 1 if:

$$P \ge P_{crit} = \frac{1}{2|\Delta q_o^*|}\left\{2|\Omega| + \left[(|\Delta q_o^*| - 2|\Omega|)^2 + \frac{8A}{T}|[h(t) - h_s]\Delta q_o^*|\right]^{1/2}\right\} - \frac{1}{2} \tag{37}$$

Proof. See appendix B.

Thus whenever $P > P_{crit}$, the fixed end-point condition has no impact on filtering performance as measured by MRCO.

It is easily verified that P_{crit} decreases as the flow imbalance increases and as the level approaches its nominal value. This observation allows us to choose P to guarantee optimal flow filtering for all disturbances above a particular magnitude that occur while the level is within some range about its nominal value.

As stated, Eq. 37 is a sufficient condition. However, as we discuss in appendix B, it is only conservative when $h(t) \ne h_s$ and a flow imbalance $\Omega(t)$ occurs which is in the direction that tends to return the level to its nominal value. For example, this is the case when the tank level is above nominal and the inlet flow rate drops. In any other situation, the condition, Eq. 37, is necessary as well as sufficient.

The fixed end-point condition is not sufficient to guarantee the realization of zero offset in P time steps. Since the on-line optimization is resolved at each sample time, the complete solution $\mathbf{q}_o^*(t)$, determined at time t, which provides zero offset in P steps, is not implemented. Instead the moving-horizon approach of implementing only the first element of $\mathbf{q}_o^*(t)$, results in the realization of the sequence $\{q_{o1}^*(t), q_{o1}^*(t+1), q_{o1}^*(t+2), \ldots\}$ (given by Eq. 35), which need not provide zero offset in P steps. This condition does however insure that there is no *steady-state* level offset.

Theorem 3. The moving horizon model predictive controller defined by Eqs. 29, 30b and 34 achieves zero steady-state level offset for constant inlet flow disturbances.

Proof. See appendix C.

Simulation experience has shown that in general the level returns to within 5% of its nominal value in between $2P$ and $2.5P$ sampling times for step inlet disturbances. For small inlet disturbances the settling time is often smaller.

The significance of these results is that for any given flow imbalance, Ω, there exists a finite P for which the moving-horizon model predictive controller with box constraints achieves the minimum possible MRCO *and* integral action. It follows that by selecting P adequately large, optimal flow filtering and integral action can be achieved for disturbances of arbitrarily small magnitude. Suboptimal filtering of small disturbances (as determined by the selection of P) is not a practical concern since these disturbances pose the least trouble for downstream equipment.

What we have achieved by introducing box constraints is to assure satisfaction of the secondary objective of integral action with no adverse impact on the primary objective of flow filtering for large disturbances. The price we pay for integral action is suboptimal filtering of small disturbances, but as we have argued, this is not significant in practice. In contrast, the addition of an integral term to an otherwise optimal controller, as in the OPC, results in suboptimal performance whenever the integral term is nonzero (essentially always). Interaction and in some cases competition between the integral and optimal terms can significantly affect filtering performance and settling time, as we will see in the examples below.

The single tuning parameter of this algorithm is the horizon length, P, which directly determines the trade-off between the incompatible objectives of good flow filtering (requiring P large) and rapid integral action (requiring P small). The appropriate value of P is determined by the characteristics of a specific implementation. The operating conditions of the upstream equipment will dictate the magnitude and frequency of expected inlet flow disturbances. The sensitivity of downstream equipment will dictate the filtering performance required for the expected disturbances. Ideally, P is selected equal to or greater than P_{crit} for the smallest disturbance for which optimal filtering is required. In general, if rapid integral action is not required (disturbances are infrequent) P should be large. If large disturbances occur frequently it may be advantageous to reduce P so that tank volume is recovered rapidly to be used to filter subsequent disturbances.

The effect of the horizon length is demonstrated in Figures 2a and 2b. The single-tank system used in this example is described in Table 1. Figure 2a shows the level response to a 50% step change in inlet flow rate (for which $P_{crit} = 14$) for $P = 5, 8, 14, 25$, and ∞. Figure 2b shows the corresponding outlet flow rates. For $P < P_{crit}$ increasing P improves flow filtering from $MRCO = 1.00$ for $P = 5$, to $MRCO = 0.36$ for $P = 14$, at the expense of settling time. For $P > P_{crit}$ no improvement in flow filtering as measured by MRCO is possible and settling time increases.

Note that as P is increased, relaxing the desired settling time,

Table 1. Example System

Cross-sectional area, A	1.0 m^2
Nominal level, h_s	1.0 m
Maximum level constraint	1.4 m
Minimum level constraint	0.6 m
Nominal outlet flow	1.0 m^3/min
Tank height	2.0 m
Outlet flow capacity	0.0—4.0 m^3/min
Sampling time, T	0.2 min

the maximum peak in outlet flow is reduced. For P infinite, the maximum peak is equal to the steady state outlet flow. This demonstrates the general result that for step inlet flow disturbances outlet flow constraints are not a problem unless rapid integral action is required (P small). Of course outlet flow capacity must be at least as large as step inlet disturbances to prevent level constraint violation at steady state.

The control algorithm resulting from box level constraints can be summarized as follows. At each sample time:

1. Update the internal model output, $\tilde{h}(t)$, based on $q_o(t-1)$ using Eq. 13
2. Evaluate the effect of disturbances on the level, $d_h(t)$, using Eq. 12
3. Evaluate the inlet flow disturbance estimate, $d(t-1|t)$, using Eq. 17
4. Evaluate the flow imbalance, $\Omega(t)$, using Eq. 19
5. Evaluate k^* using Eq. 21
6. Evaluate Δq_o^* using Eq. 20

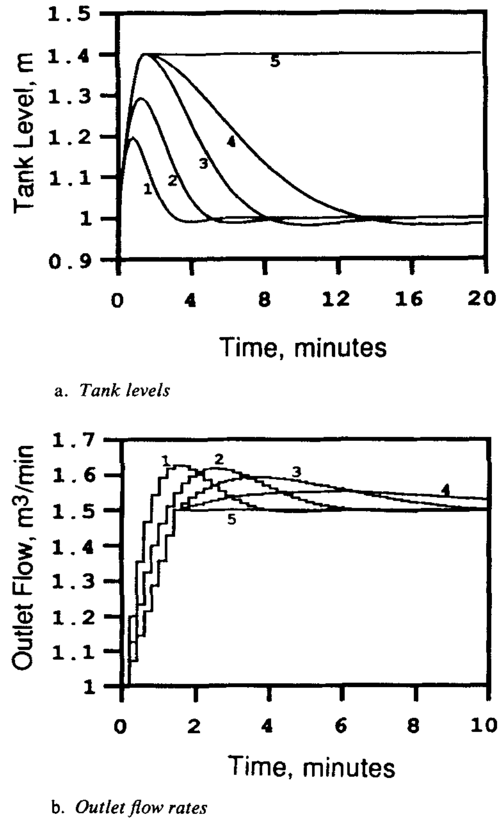


Figure 2. Effects of a 50% inlet flow disturbance.
Model predictive controller with box constraints: 1. $P = 5$; 2. $P = 8$; 3. $P = 14$; 4. $P = 25$; 5. $P = \infty$

7. Evaluate Δq_o^0 using Eq. 36
8. Change the tank outlet flow by Δq_o^* or Δq_o^0, depending on which has the larger magnitude, Eq. 35

When outlet flow constraints are not present and internal stability problems are not a practical concern (see below), this algorithm will provide optimal flow filtering with integral action. In the more general case, an analytical solution to the optimization problem is not available. In the next section we show how the optimization can be recast as a linear program (for which a number of numerical solution techniques are available) to be solved on-line.

Other level constraints

The linear program formulation outlined in the next section allows very general specification of future level constraints. For example, adopting the approach of Cutler (1982), the set of admissible predicted levels might be selected so that the future level lies within a target area, centered at the nominal level, with a magnitude that decreases into the future. A fixed end-point condition included in the definition of the target area insures zero steady-state offset. In general these more restrictive level constraints result in poorer flow filtering and faster integral action relative to box constraints. Since the moving-horizon implementation does not guarantee that the level will not leave the target area, it is unlikely that such constraint sets offer any additional advantages.

Formulation as a Linear Program

With outlet flow constraints the closed form solutions, Eqs. 32 and 35, are not valid. In this case Eqs. 29–30 must be solved on-line at each sample time. It remains to be shown how this problem can be recast as a linear program.

Following the standard approach for solving Chebyshev approximation problems via linear programming, we define:

$$\mu^*(\boldsymbol{q}_o) = \|\boldsymbol{R}\boldsymbol{q}_o - e_1 q_o(t-1)\|_\infty \tag{38}$$

Any μ that satisfies

$$\begin{aligned} -\mathbf{1}\mu &\le -\boldsymbol{R}\boldsymbol{q}_o + e_1 q_o(t-1) \\ -\mathbf{1}\mu &\le \boldsymbol{R}\boldsymbol{q}_o - e_1 q_o(t-1) \end{aligned} \tag{39}$$

represents an upper bound on μ^*. The task is now to find $\boldsymbol{q}_o$ and μ that satisfy Eqs. 30 and 39 and simultaneously minimize $f(\boldsymbol{q}_o, \mu) = \mu$. This problem is easily formulated as:

$$\min_{\mu, \boldsymbol{q}_o} \mu \tag{40}$$

Subject to:

$$\begin{aligned} \boldsymbol{R}\boldsymbol{q}_o - \mathbf{1}\mu &\le e_1 q_o(t-1) \\ -\boldsymbol{R}\boldsymbol{q}_o - \mathbf{1}\mu &\le -e_1 q_o(t-1) \\ -\boldsymbol{q}_o + \boldsymbol{\alpha} &\le \mathbf{0} \\ \boldsymbol{q}_o &\le \boldsymbol{\beta} \\ \boldsymbol{H}\boldsymbol{q}_o &\le -\boldsymbol{\gamma} + \boldsymbol{n}\frac{T}{A} d(t+k|t) + \mathbf{1}h(t) \\ -\boldsymbol{H}\boldsymbol{q}_o &\le \boldsymbol{\delta} - \boldsymbol{n}\frac{T}{A} d(t+k|t) - \mathbf{1}h(t) \end{aligned} \tag{41}$$

Defining

$$\bar{q}_o = q_o - \alpha \tag{42}$$

and

$$c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} \mu \\ \bar{q}_o \end{pmatrix} \quad A = \begin{pmatrix} -1 & R \\ -1 & -R \\ 0 & I \\ 0 & H \\ 0 & -H \end{pmatrix}$$

$$b = \begin{pmatrix} e_1 q_o(t-1) - R\alpha \\ -e_1 q_o(t-1) + R\alpha \\ \beta - \alpha \\ -\gamma + n\dfrac{T}{A} d(t+k|t) + \mathbf{1}h(t) - H\alpha \\ \delta - n\dfrac{T}{A} d(t+k|t) - \mathbf{1}h(t) + H\alpha \end{pmatrix}$$

we obtain the linear program in standard form,

$$\min_x c^T x$$

subject to:

$$Ax \le b$$
$$x \ge \mathbf{0} \tag{43}$$

Using Eq. 42, we have employed the nonnegativity condition $x \ge \mathbf{0}$ to enforce the lower bound on q_o. The resulting linear program involves $P + 1$ variables and $5P$ constraints (not counting the nonnegativity constraints). As discussed by Campo and Morari (1986), it is more efficient computationally to solve the dual program:

$$\min_y - b^T y$$

subject to:

$$-A^T y \le c$$
$$y \ge \mathbf{0} \tag{44}$$

which involves $5P$ variables and $P + 1$ constraints.

Implementation

Before implementing this model predictive scheme we must consider internal stability. It is well known that controllers implemented in the internal model control structure are not internally stable when the plant is not stable. The proposed model predictive control algorithm is a special case of such an implementation and therefore deserves further analysis. For a general discussion of internal stability, the interested reader is refered to Morari and Zafiriou (1989).

The following analysis is based on Figure 3, where we have

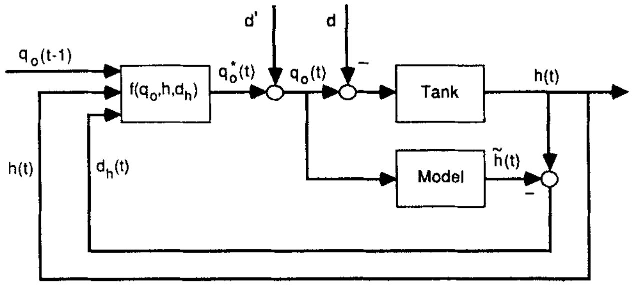


Figure 3. Model predictive structure.

represented the on-line optimization as a mapping, $f[q_o(t-1), h(t), d_h(t)]$, which takes current values of the manipulated and controlled variables, and the effect of disturbances on the level, and yields a new optimal value for the manipulated variable, $q_o^*(t)$. At steady state we have, $q_o(t-1) = h(t) = d_h(t) = 0$, and the optimal solution is $q_o^*(t) = 0$.

Suppose a step disturbance enters at d' immediately before the optimal solution, $q_o^*(t)$, is implemented, that is, at time t_-. At the next sampling time we have, with perfect modeling:

$$h(t+1) = \tilde{h}(t+1) = \frac{Td'}{A} \tag{45}$$

so that

$$d_h(t+1) = 0 \tag{46}$$

The predicted level, with future outlet flows zero, is:

$$h(t+k|t) = h(t+1) = \frac{Td'}{A} \quad \forall k = 1, 2, \ldots, P \tag{47}$$

If this predicted future level is feasible for all $0 < k < P$, (for example if constant level constraints with $h_{max} > Td'/A$ have been specified) then the solution $q_o(t+1) = 0$, is feasible (and obviously optimal). Similarly, the controller will take no action in response to the disturbance d' at subsequent sample times, while the actual level, given by

$$h(t+k) = \frac{kTd'}{A} \tag{48}$$

integrates away from its nominal value. In this sense, the algorithm is internally unstable. Although not identified as such, this internal instability was observed by McDonald et al. as drifting in the level as a result of constant bias betwen the inlet and outlet flow rate measurements of their optimal predictive controller (OPC). In fact any disturbance at d' will result in drifting of the level.

Disturbances occur at d' whenever the output of the controller is not equal to the actual value implemented on the process and provided to the internal model. It is common practice to readback the valve position from its actuator so that the value implemented on the process can be supplied to the internal model. Quite often the manipulated variable value realized by the actuator differs significantly from the value commanded. (It is this situation that makes readback necessary.) This results in significant differences between the controller output and the readback signal provided to the internal model. Any such difference is effectively a disturbance at d' that will not be compensated for by the model predictive controller. Thus readback should not be used in the implementation of a model predictive controller when the plant includes an integrator.

An example of the unstable response resulting from a disturbance d' when readback is used is shown in Figure 4 (curve *1*). Here we have simulated the system described in Table 1, using the model predictive controller with constant level constraints, and included a constant disturbance, $d' = 0.1\ m^3/min$. This disturbance could arise from a bias in the outlet flow actuator resulting in an outlet flow 0.1 m^3/min greater than commanded. Since $d_h(t)$ remains zero for all t, the controller takes no action as the level falls by 0.02 m at each sample time, eventually draining the tank completely.

When readback is not used, disturbances d' can arise from algorithmic (round-off) errors in the implementation of the optimal solution. In practice these errors would be expected to be small, and since for integrators the growth of the instability is only linear, it is reasonable that in practice these errors could take a very long time to have a significant impact on the system.

When level constraints include a fixed end-point condition, $q_o(t + 1) = 0$ is not feasible since $h(t + P|t) = Ad'/T \neq 0$. In this case, a nonzero response to disturbances d' is provided. However these disturbances still result in undesired drifting in the tank level. While it is reasonable to suggest that if readback is not used a direct implementation of the MPC scheme might be successful in practice, in the next section we discuss an implementation of the MPC controller that is guaranteed to be internally stable.

Stabilizing Embedded Feedback

Assuming perfect modeling, internal stability of the model predictive control scheme is guaranteed if the plant is stable (Morari and Zafiriou, 1989). For unstable plants, we can first stabilize the plant with an internal feedback loop, as in Figure 5a. The embedded controller, $K(s)$, is chosen to stabilize the plant, $P(s) = -1/As$. To apply model predictive control, we treat the embedded system as a stable 1 × 2 plant, P^*, with the (unmeasured) disturbance, d, and a setpoint, r, as inputs, and the tank level, h, as output, as shown in Figure 5b. The appropriate transfer matrices are:

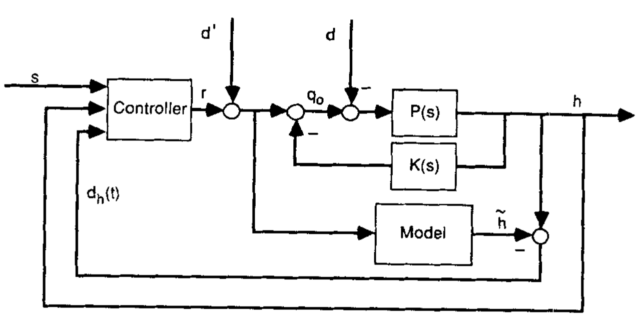


Figure 5a. Model predictive structure with embedded feedback.

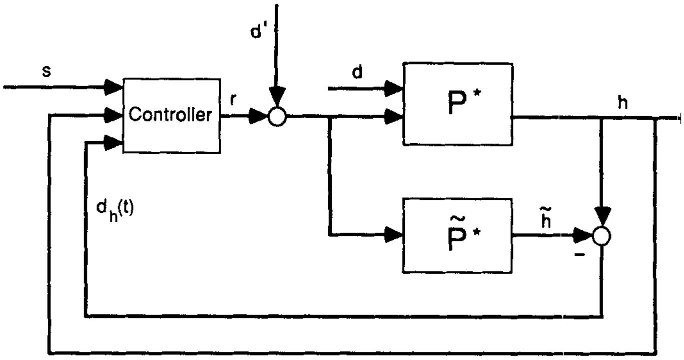


Figure 5b. Equivalent representation of Figure 5a for internal stability analysis.

$$P^* = [(I + PK)^{-1}P \ -\ (I + PK)^{-1}P] \tag{49}$$

$$\tilde{P}^* = [(I + \tilde{P}K)^{-1}\tilde{P}] \tag{50}$$

If the controller, $K(s)$, internally stabilizes the plant then P^* is (necessarily) stable. It is straightforward then to use the on-line optimization of model predictive control to determine set points for the embedded system that provide the desired performance subject to constraints on $q_o(t + k)$ and $h(t + k|t)$. The stable response to disturbances, d', is now given by:

$$h(s) = (I + PK)^{-1}Pd'(s) \tag{51}$$

The design of $K(s)$ introduces no theoretical limitation on the achievable input/output properties of the overall system (i.e., the transfer functions $h(s)/d(s)$ and $h(s)/s(s)$ of Figure 5) if $K(s)$ is stable (Zames, 1981). Since a stable controller (e.g., $K(s) = K_c$) is adequate to stabilize $P(s)$, this restriction poses no problem in this application.

By proper selection of $K(s)$ we can assure that signals entering at d' are attenuated over a desired frequency range. In particular, it is clear from Eq. 51 that the steady-state attenuation of step disturbances d' is inversely proportional to the steady-state gain of $K(s)$.

Repeating the actuator bias example of the previous section with discrete-time embedded feedback given by $K(z) = z/z - 0.6$ results in the stable response shown in Figure 4 (curve *2*). Although the model predictive controller takes no action, the embedded feedback prevents the level from violating its con-

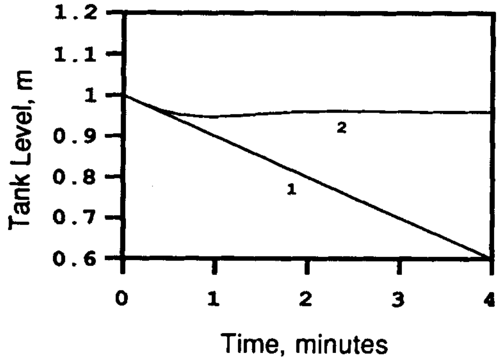


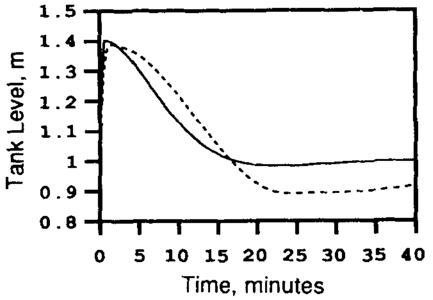
Figure 4. Tank level changes resulting from actuator bias.
1. Without embedded feedback
2. With embedded feedback

straint. As expected from Eq. 51, the step disturbance d' of magnitude 0.1 m^3/min produces a steady-state offset in level of 0.04 m.

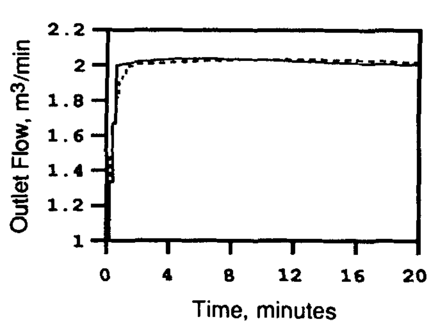
With embedded feedback, the decision variable of the on-line optimization is the set point to the embedded controller, r. Thus, the nonnegativity conditions of the linear program cannot be used to enforce bounds on the outlet flow rate as in Eq. 42. This results in an increase in the size of the linear program that must be solved on-line. The program corresponding to the model predictive algorithm with embedded feedback involves $2P + 1$ variables and $6P$ constraints. The dual program (which would be solved in practice) involves $6P$ variables and $2P + 1$ constraints.

Examples

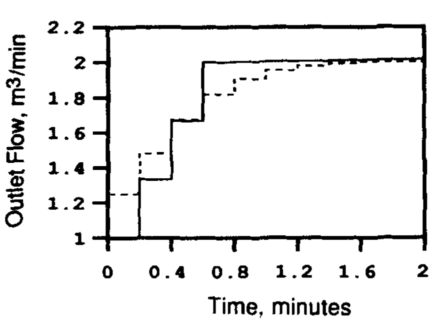
A simulation study was carried out to demonstrate the performance of the model predictive scheme relative to the discrete infinite-horizon and optimal predictive controllers. The example system proposed by Cheung and Luyben (1979), and adopted by McDonald et al. (1986) is used here; specifications are given in Table 1.



a. *Tank levels*



b. *Outlet flow rates*



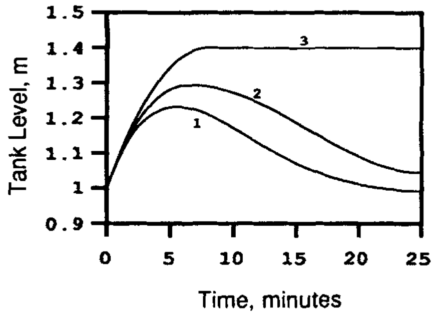
c. *Outlet flow rates, detail*

Figure 6. Effects of a 100% inlet flow disturbance.
— Model predictive controller with box constraints
--- OPC

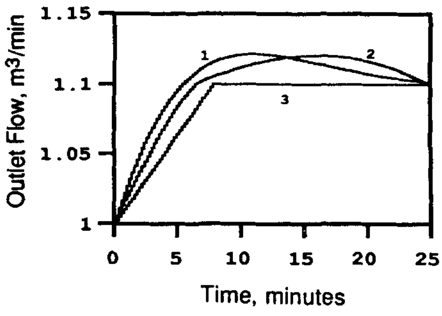
Figures 6a, 6b, and 6c show the level and outlet flows corresponding to an inlet step disturbance of 100% of the nominal flow for the model predictive with box constraints and OPC schemes. The proportional gain and integral reset time for the OPC were 0.046 and 3.0, as suggested by McDonald et al. In order to obtain optimal filtering for inlet flow disturbances larger than 25% of the nominal inlet flow, a horizon, P, of 35 sample times (7 min) was chosen for the model predictive controller. For the 100% disturbance, $k^* = 2$, and $P_{crit} = 6$. MRCO is 1.67 for the model predictive controller and 1.25 for the OPC. The OPC is able to achieve lower MRCO since it uses inlet flow measurements and can adjust the outlet flow immediately, while the model predictive algorithm requires 1 sample time to infer the flow disturbance from level measurements. Note that since $P > P_{crit}$, the fixed end-point condition does not affect MRCO. The model predictive controller returns the level to within 5% of nominal in 13.0 min, the OPC requires 46.8 min.

Figures 7a and 7b show the level and outlet flows corresponding to an inlet step disturbance of 10% for the model predictive ($P = 35$), discrete infinite-horizon, and OPC schemes. For this disturbance, $k^* = 38$ and $P_{crit} = 93$. As expected, the model predictive scheme results in higher MRCO (0.0286) than the OPC (0.0175) for this disturbance. At the expense of increased settling time, P could be made greater than P_{crit} to achieve the best possible discrete-time flow filtering ($MRCO = 0.0128$) realized by the discrete infinite-horizon controller.

McDonald et al. suggest that the integral reset time of the OPC can be adjusted to achieve a desired settling time. Figures 8a and 8b show the effect of decreasing τ_i for the OPC to improve the settling time in response to a 50% step inlet disturbance. As shown in Figure 8a, the model predictive controller, with $P = 35$, returns the level to within 5% of nominal in 13.8 min with a MRCO of 0.358. To obtain this same settling time

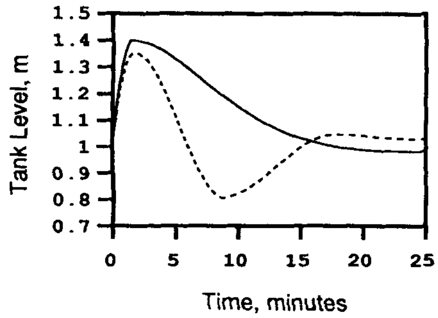


a. *Tank levels*

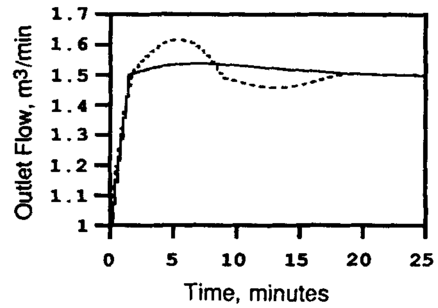


b. *Outlet flow rates*

Figure 7. Effects of a 10% inlet flow disturbance.
Model predictive controller with box constraints: 1. $P = 35$; 2. OPC; 3. $P = \infty$



a. *Tank levels*



b. *Outlet flow rates*

Figure 8. Effects of a 50% inlet flow disturbance.

— Model predictive controller with box constraints, $P = 35$

--- OPC tuned to achieve equivalent settling time

with the OPC (with $K_c = 0.046$) an integral reset time of 0.32 min was required, resulting in a MRCO of 0.333. (Settling time and MRCO for the OPC with the tuning parameters suggested by McDonald et al. were 60.0 min and 0.327.) Again, the superior MRCO performance of the OPC comes from the use of feedforward measurements. However the OPC has a much greater outlet flow overshoot and the level and outlet flow responses are much more oscillatory than for the model predictive controller. We quantify this latter observation with the following definition.

Defining

$$MRCO(t_o) = \sup_{\substack{t \in (t_o, \infty) \\ t \neq t'}} \left| \frac{q_o(t) - q_o(t')}{t - t'} \right| \tag{52}$$

we introduce a generalization of MRCO. $MRCO(t_o)$ is simply a measure of filtering performance for times after t_o. If we let t_o be the time at which the flow imbalance is offset (t^* or k^*), we can use Eq. 52 as a performance measure for the time period in which the controller returns the level to its nominal value. In the previous example, Figure 8, $t_o = 1.6$ for the OPC and $t_o = 1.4$ for the model predictive controller. $MCRO(1.6)$ for the OPC is 0.0728 while $MRCO(1.4)$ for the model predictive controller is 0.0158. After the flow imbalance has been offset, the model predictive controller returns the level to its nominal value with outlet flow changes one-fifth as large as the OPC. The greater oscillation and flow overshoot demonstrated by the OPC when relatively small settling times are required are a result of noncooperative interactions between the optimal and integral terms in Eq. 6. To achieve rapid settling the integral term must be made significant, and this negatively affects flow filtering performance.

Conclusions

The discrete-time analog of the optimal averaging control problem has been defined and solved. This solution provides the minimum achievable MRCO consistent with constant level constraints. The use of the internal model control structure insures that flow disturbances are offset optimally without requiring feedforward measurements.

Insight gained from the solution of the discrete-time, infinite-horizon problem motivates the formulation of a finite-horizon problem using the on-line optimization and moving-horizon ideas of model predictive control (MPC). An analytical solution to the finite moving-horizon problem allows us to develop conditions under which the finite-horizon solution recovers the infinite-horizon solution. Specifically, it is shown that this is the case for large disturbances for which optimal flow filtering is most critical.

Introducing a fixed end-point condition (box level constraints) we show that integral action can be obtained without sacrificing optimal flow filtering for disturbances above a specified threshold magnitude. This formulation is an attractive alternative to control schemes that add integral action to an otherwise optimal controller in an *ad hoc* fashion. Additionally, a single tuning parameter, the horizon length, simply and directly effects the trade-off between flow filtering and rapid integral action.

The new surge tank level controller is formulated as a model predictive control problem involving the solution of a linear program at each sample time. This application demonstrates the flexibility of MPC and the relative ease with which it can be applied to control problems with nontraditional objectives. The use of an embedded (local) stabilizing controller insures internal stability of the internal model structure (even though the plant, a pure integrator, is not asymptotically stable).

Acknowledgment

Partial financial support for this work provided by the National Science Foundation is gratefully acknowledged.

Notation

d = inlet flow disturbance
h = tank level
K = the set of nonnegative integers $\{0, 1, 2, \ldots\}$
K_c = proportional gain
P = model predictive control horizon length
q = flow rate
t = time
T = sampling time
τ_i = integral reset time

Subscripts

i = inlet
max = maximum allowed value
min = minimum allowed value
o = outlet
s = steady state

Superscripts

$\tilde{}$ = plant model, or value determined by the plant model
$*$ = optimal value

Appendix A: Proof of Theorem 1

The proof of theorem 1 is straightforward but tedious. For simplicity many details have been omitted. Throughout the proof we assume that $\Omega(t) \geq 0$; the results for $\Omega(t) \leq 0$ follow in a parallel fashion.

Forward direction ($\Rightarrow$):

We first show that

$$q_o^*(t+k) = \begin{cases} q_o(t-1) + (k+1)\Delta q_o^* & k \in [0, k^*) \\ d(t+k|t) & k \geq k^* \end{cases}$$

is a feasible solution. By direct substitution into Eq. 18 it is easy to show that, for $k < k^*$,

$$h^*(t+k|t) = \begin{cases} h(t) + \dfrac{kT}{A}\left\{\Omega(t)\left(1 - \dfrac{k+1}{k^*+1}\right) \right. \\ \left. + \dfrac{k+1}{k^*(k^*+1)} \dfrac{A[h_{max} - h(t)]}{T}\right\} \end{cases}$$

and, for $k \geq k^*$,

$$h^*(t+k|t) = h_{max}$$

Clearly $h^*(t+k|t)$ is nondecreasing in k for $k \leq k^*$ and $h^*(t+k^*|t) = h_{max}$ so that $q_o^*(t+k)$ is feasible. This implies condition 3 of theorem 1.

We now show that all solutions satisfying condition 3 but not condition 1 are infeasible. For any such solution, $\hat{q}_o$ there must exist some $\hat{k} < k^*$ such that

$$\hat{q}_o(t+k) < q_o^*(t+k) \quad \forall k \in [\hat{k}, k^*)$$

It follows that the corresponding predicted levels must obey

$$\hat{h}(t+k+1|t) > h^*(t+k+1|t) \quad \forall k \in [\hat{k}, k^*)$$

but $h^*(t+k^*|t) = h_{max}$, so that $\hat{h}(t+k^*|t) > h_{max}$, which implies that $\hat{q}_o$ is infeasible. Thus condition 1 is established.

We now show that all feasible solutions satisfying condition 1 must also satisfy condition 2. From Eq. 18,

$$h(t+k^*+j|t) = h(t+k^*|t) + \frac{T}{A}\sum_{i=0}^{j} d(t+k|t) - q_o(t+k^*+i) \quad (A1)$$

As we have seen, $h(t+k^*|t) = h_{lim}$ for any solution satisfying condition 1, so we have from Eq. A1:

$$h_{min} \leq h(t+k^*+j|t) \leq h_{max}$$

$$\Longleftrightarrow$$

$$h_{min} - h_{lim} \leq \frac{T}{A}\sum_{i=0}^{j} d(t+k|t) - q_o(t+k^*+i) \leq h_{max} - h_{lim}$$

$$\Longleftrightarrow$$

$$h_{max} - h_{lim} \geq \frac{T}{A}\sum_{k=k^*}^{j} d(t+k|t) - q_o(t+k) \geq h_{min} - h_{lim} \quad \forall j \geq k^*$$

Thus for feasibility we must have condition 2.

To show the reverse direction ($\Leftarrow$) we assume conditions 1, 2, and 3 hold for some $\hat{q}_o(t+k)$, $k \in K$. As we showed above, conditions 1 and 2 are necessary and sufficient for feasibility, given condition 3. Thus feasibility of $\hat{q}_o$ is established. Since condition 1 implies

$$\min_{\hat{q}_o(t+k)} \max_{k \in K} |\hat{q}_o(t+k) - \hat{q}_o(t+k-1)| \geq \Delta q_o^*$$

any $\hat{q}_o$ satisfying condition 3 must be optimal and we have completed the proof.

Appendix B: Proof of Theorem 2

In order to satisfy condition 1 of theorem 1 for a nonzero flow imbalance at time t_0 we must implement Δq_o^* $\forall t < t_0 + k^*$. This is guaranteed by

$$|\Delta q_o^*(t_0)| > |\Delta q_o^0(t_0)| \quad (B1)$$

since $|\Delta q_o^0|$ decreases monotonically while $|\Delta q_o^*|$ is constant for $t \leq t_0 + k^*$. Thus $|\Delta q_o^0(t)|$ remains less than $|\Delta q_o^*(t)|$ and Δq_o^* is implemented until $t = t_0 + k^*$. For $t \geq t + k^*$, $\Delta q_o^0(t+k^*)$ is implemented, guaranteeing feasibility (condition 2 of theorem 1). Condition 3 is satisfied by Eq. B1 since $|\Delta q_o^0|$ is bounded by $|\Delta q_o^*(t_0)|$ for all time. From Eqs. 20 and 36 it is straightforward to verify that Eq. B1 is equivalent to

$$P^2 + P - \left|\frac{2TP\Omega + 2A[h(t) - h_s]}{T\Delta q_o^*}\right| \geq 0 \quad (B2)$$

Since

$$|2TP\Omega| + |2A[h(t) - h_s]| \geq |2TP\Omega + 2A[h(t) - h_s]| \quad (B3)$$

Eq. B2 is implied by

$$P^2 + P\left[1 - \frac{2|\Omega|}{|\Delta q_o^*|}\right] - \frac{2A|h(t) - h_s|}{T|\Delta q_o^*|} \geq 0 \quad (B4)$$

By direct application of the quadratic formula it can be verified that Eq. 37 is equivalent to Eq. B4.

Note that the use of the triangle inequality, Eq. B3, results in a sufficient condition on P. However in practice we are certain to encounter situations where both terms of Eq. B3 have the same sign so that equality holds and the sufficient condition is necessary as well. For example, a step disturbance from the nominal steady-state results in both terms of Eq. B3 having the same sign. In fact we do not have equality in Eq. B3 only when the level is not at its steady state value and an imbalance occurs that is in the "good" direction (i.e., tending to return the level to nominal).

Appendix C: Proof of Theorem 3.

At steady state we must have $\Delta h = \Delta q_o = 0$. Using Eq. 35, $\Delta q_o = 0$ implies that $\Delta q_o^* = \Delta q_o^0 = 0$ at steady state. From Eq. 36, $\Delta q_o^0 = 0$ implies

$$0 = \frac{2\Omega}{P+1} + \frac{2A[h(k) - h_s]}{TP(P+1)}$$

Clearly, $\Omega = 0$ at steady state since if there is a nonzero flow imbalance we cannot have $\Delta h = 0$. Thus

$$\frac{2A[h(k) - h_s]}{TP(P+1)} \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty$$

which implies

$$h(k) \longrightarrow h_s \quad \text{as} \quad k \longrightarrow \infty$$

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